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A projector formulation of the Galilean covariant Duffin–Kemmer–Petiau field

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Abstract
We construct in this paper a projector formulation of the Galilean covariant Duffin–Kemmer–Petiau field in five dimensions. Such an approach allows us to select the scalar and vector sectors of the theory through the use of the appropriate operators. As an application, we study a non-minimal coupling associated to the DKP harmonic oscillator, naturally in the non-relativistic regime and with the selection of the spin-0 sector in a general representation. We also discuss the local gauge invariance and the anomalous term which appear in the wave equations due to the minimal coupling. In both questions, our results were carried out as in the relativistic case, i.e. the consistent local gauge transformations can be obtained through the choice of the right form of the non-relativistic DKP field.

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1. Introduction

The first order wave equation for the relativistic scalar and vector fields was proposed in [1] by the so-called Duffin–Kemmer–Petiau (DKP) equation. The historical details of its original version can be obtained in [2, 3] and its extension for higher spin theories is presented in [4]. This formulation has been studied in many different contexts: in QCD at large and short distances [5], Einstein gravity [6], spacetime with torsion [7], Bose–Einstein condensation [8] and field theory at finite temperature [9].

In particular, in [10] were discussed aspects concerning the minimal electromagnetic coupling in the DKP theory, with the right analysis of the physical components of DKP field being obtained, circumverting the apparent difference between the interaction terms in DKP and Klein–Gordon Lagrangians, as well as the presence of an anomalous term.

Turning to the non-relativistic context, first-order wave equations were also constructed through a Galilean covariant formulation of the Bhabha formalism, in such a way that for spins 0 and 1 the non-relativistic DKP equation for scalar and Proca fields are [11, 12] reproduced.

In general lines, the Galilean covariance approach above-mentioned follows the usual tools of the relativistic theories with Lorentz covariance. The starting point is the Galilean invariance of the theory, with the spacetime being extended by the addition of an extra coordinate using its immersion in a de Sitter 4 + 1 space. It is done defining the five-dimensional manifold with the coordinates

$$x^\mu = (x^1, x^2, x^3, x^4, x^5) = (\mathbf{x}, t, s), \tag{1}$$

that transforms by

$$\begin{aligned} \mathbf{x}' &= R\mathbf{x} + \mathbf{v}t + a, \\ t' &= t + b, \\ s' &= s + (R\mathbf{x}) \cdot \mathbf{v} + \tfrac{1}{2}\mathbf{v}^2 t. \end{aligned} \tag{2}$$

This transformation leaves invariant the scalar product $g_{\mu\nu} dx^\mu dx^\nu$, where $g_{\mu\nu}$ is the Galilean metric given by

$$(g^{\mu\nu}) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & 0 \end{pmatrix}. \tag{3}$$

In [13–17] there are other examples in which this Galilean covariant formulation is applied.

However, despite the consistent construction of the Galilean covariant formulation of DKP theory in [11, 12], some questions remain when one considers the interaction of the DKP field with the Galilean eletromagnetic field, performed via minimal coupling. Similarly to [10], it is expected from this interaction an anomalous term in the motion equation that has not a clear physical meaning, and also a linear term with the field A_μ. Thus, motivated in [10], the aim of the present work is to show that both situations can be correctly understood when the correct form of the coupling is performed.

This paper is organized as follows. In the second section we obtain operators that will be useful tools to select the scalar and vector sectors of the theory and present its application to the scalar boson oscillator case. The section three is dedicated to treat the central problems: the gauge invariance of the Galilean DKP theory and the anomalous term that appears in the wave equation when the coupling is performed.

2. The Galilean covariant theory and projector formulation

We start with the Lagrangian density for the Galilean covariant free DKP field,

$$\mathcal{L} = \frac{1}{2}\overline{\Psi}\beta^\mu \partial_\mu \Psi - \frac{1}{2}(\partial_\mu \overline{\Psi})\beta^\mu \Psi + k\overline{\Psi}\Psi, \tag{4}$$

where the index μ runs from 1 to 5 and k is a constant; the matrices are such that satisfy the algebra

$$\beta^\mu \beta^\nu \beta^\rho + \beta^\rho \beta^\nu \beta^\mu = g^{\mu\nu}\beta^\rho + g^{\nu\rho}\beta^\mu. \tag{5}$$

We have also introduced in equation (4) the adjoint spinor, given by $\overline{\Psi} = \Psi^\dagger \eta$, with $\eta = (\beta^4 + \beta^5)^2 + \mathbf{1}$. It is relevant to note that we work with the choice $(\beta^i)^\dagger = \beta^i$, $(\beta^4)^\dagger = -\beta^5$ and $(\beta^5)^\dagger = -\beta^4$.

The equation of motion obtained from Lagrangian (4), henceforth called as DKP equation, is given by

$$(\beta^\mu \partial_\mu + k)\Psi = 0. \tag{6}$$

It is worth mentioning that to preserve the invariance of the DKP equation and DKP Lagrangian under Galilei transformations, the coordinates x^μ, the field Ψ and the matrices β^μ must transform according to

$$\begin{aligned} x'^\mu &= \Lambda^\mu{}_\nu x^\nu, \\ \Psi'(x) &= U(\Lambda)\Psi(\Lambda^{-1}x), \\ U^{-1}(\Lambda)\beta^\mu U(\Lambda) &= \Lambda^\mu{}_\nu \beta^\nu. \end{aligned} \tag{7}$$

If the Galilei transformations are infinitesimal, we have $\Lambda^{\mu\nu} = g^{\mu\nu} + w^{\mu\nu}$, with $w^{\mu\nu} = -w^{\nu\mu}$ and

$$U = 1 + \tfrac{1}{2} w^{\mu\nu} S_{\mu\nu} \tag{8}$$

with $S_{\mu\nu} = [\beta_\mu, \beta_\nu]$.

Note that the multiplication of the DKP equation (6) by the operator $\partial_\alpha \beta^\alpha \beta^\nu$ from the left, and after contracting it with ∂_ν, yields

$$(\partial^\mu \partial_\mu + k^2)\Psi = 0. \tag{9}$$

Thus, each component of the spinor Ψ must obey a Klein-Gordon-like equation. However, in our case equation (9) must be understood as a Galilean covariant version of the Schrödinger equation, which is the right-field equation of the non-relativistic regime [17].

Now we discuss the significance of the field Ψ and equations (6) and (9). They can be better perceived after the analysis of the irreducible representations of the symmetry group under which the DKP theory is invariant. In a similar way to the non-relativistic Bhabha formulation [4], the non-relativistic DKP formulation can be obtained from the irreducible representations of the Lie algebra $so(5, 1)$, which is an extension of the algebra $so(4, 1)$ (the algebra of the Galilean covariant formulation) including the five generators β^μ.

Then, in the scenario above-mentioned, the dimensions of the irreducible representations for the DKP formulation are 6 and 15, which describe particles with spins 0 and 1, respectively. Therefore, the DKP theory presents two sectors, scalar and vector, that have to be selected with the appropriated operators to get the physical meaning of the field Ψ.

These operators can be constructed by using a general representation of β^μ matrices, as in the usual relativistic way [10, 18]. For the scalar sector, they are given by

$$P = -\tfrac{1}{2}(\beta^4 + \beta^5)^2(\beta^1)^2(\beta^2)^2(\beta^3)^2, \tag{10}$$

and

$$P^\mu = P\beta^\mu. \tag{11}$$

From these definitions, we can show the identities

$$P^2 = P, \qquad P^\mu \beta^\nu = P g^{\mu\nu}. \tag{12}$$

The consequences of the action of these operators on the DKP field are as follows: $PU\Psi = P\Psi$ and $P^\mu U\Psi = P^\mu\Psi + w^\mu{}_\nu P^\nu\Psi$, which show that $P\Psi$ transforms like a scalar and $P^\mu U\Psi$ like a vector. In addition, applying these identities to the DKP equation (6), we arrive to

$$(\partial^\mu \partial_\mu + k^2)(P\Psi) = 0. \tag{13}$$

Thus, each component of $P\Psi$ can be interpreted as a scalar field that obeys to the Galilei covariant Schrödinger wave equation.

Let us construct now the operator that selects the vector sector in the DKP theory. For this we define the operators

$$R^{\mu} = (\beta^1)^2(\beta^2)^2(\beta^3)^2[\beta^{\mu}(\beta^4+\beta^5) - g^{\mu 4} - g^{\mu 5}], \tag{14}$$

and

$$R^{\mu\nu} = R^{\mu}\beta^{\nu}. \tag{15}$$

These operators have the properties

$$\begin{aligned} &R^{\mu\nu} = -R^{\nu\mu},\\ &R^{\mu\nu}\beta^{\alpha} = g^{\nu\alpha}R^{\mu} - g^{\mu\alpha}R^{\nu},\\ &R^{\mu}S^{\nu\alpha} = g^{\mu\nu}R^{\alpha} - g^{\mu\alpha}R^{\nu},\\ &R^{\mu\nu}S^{\alpha\rho} = g^{\mu\rho}R^{\nu\alpha} - g^{\mu\alpha}R^{\nu\rho} + g^{\nu\alpha}R^{\mu\rho} - g^{\nu\rho}R^{\mu\alpha}. \end{aligned} \tag{16}$$

From these properties it can be shown that

$$\begin{aligned} &R^{\mu}U\Psi = R^{\mu}\Psi + w^{\mu}_{\alpha}R^{\alpha}\Psi,\\ &R^{\mu\nu}U\Psi = R^{\mu\nu}\Psi + w^{\nu}_{\alpha}R^{\mu\alpha}\Psi + w^{\mu}_{\alpha}R^{\alpha\nu}\Psi. \end{aligned} \tag{17}$$

It shows that the operator $R^{\mu}\Psi$ transforms like a vector and $R^{\mu\nu}\Psi$ as a tensor. Applying these results on the DKP equation (6) we obtain

$$\partial_{\nu}G^{\nu\mu} + k^2R^{\mu}\Psi = 0 \tag{18}$$

where

$$G^{\mu\nu} = \partial^{\mu}R^{\nu}\Psi - \partial^{\nu}R^{\mu}\Psi. \tag{19}$$

The equation (18) implies

$$(\partial_{\nu}\partial^{\nu} + k^2)(R^{\mu}\Psi) = 0, \qquad \partial_{\mu}R^{\mu}\Psi = 0, \tag{20}$$

which is the manifestly covariant Schrödinger wave equation that describes the non-relativistic vector field. On the other words, similarly to the scalar case, $R^{\mu}\Psi$ selects the vector sector of the DKP field.

Up to now all the results were derived in a general framework, without the use of a particular representation of the β matrices. We can explicit a specific selection by considering the field Ψ written as

$$\Psi^T = (\Psi_1, \Psi_2, \Psi_3, \Psi_4, \Psi_5, \Psi_6)\,, \tag{21}$$

and choosing a six-dimensional representation of the β matrices (shown in the appendix) to obtain

$$P\Psi = \begin{pmatrix} 0_{5\times 1}\\ \Psi_6 \end{pmatrix}, \qquad P_{\mu}\Psi = \begin{pmatrix} 0_{5\times 1}\\ \Psi_{\mu} \end{pmatrix}, \tag{22}$$

where the operators P and P^{μ} defined in equations (10) and (11) have been used. Moreover, the use of these results and application of the operator P^{μ} to the DKP equation (6) imply

$$\Psi = \begin{pmatrix} -\frac{1}{\sqrt{k}}\partial_{\mu}\Phi\\ \sqrt{k}\Phi \end{pmatrix}. \tag{23}$$

Now, the equation (22) can be rewritten as

$$P\Psi \begin{pmatrix} 0_{5\times 1}\\ \sqrt{k}\Phi \end{pmatrix}, \qquad P^{\mu}\Psi = -\frac{1}{\sqrt{k}}\begin{pmatrix} 0_{5\times 1}\\ \partial^{\mu}\Phi \end{pmatrix} \tag{24}$$

and the field Φ obeys to the non- relativistic Schrödinger wave equation

$$(\partial_\mu \partial^\mu + k^2)\Phi = 0. \tag{25}$$

A similar result can be obtained for the spin 1 case, taking into account that in this context we have to use the 15-dimensional representation of the matrices β and the operators R^μ and $R^{\mu\nu}$.

Let us describe an application for the operator spin 0, using a general six-dimensional representation, which is the non-minimal coupling used in [4, 11], that is

$$\mathbf{p} \rightarrow \mathbf{p} + \mathrm{i} w \eta \mathbf{r}. \tag{26}$$

This coupling is performed using the DKP equation in the momenta representation, yielding

$$(\beta^\mu p_\mu + \mathrm{i} w \beta^i \eta r_i - \mathrm{i}k)\psi = 0. \tag{27}$$

Applying on this equation the operators P, P^μ and using the identities $P^i \eta = P^i$ and $P\eta = -P$ we have

$$E(P\psi) = \left(\frac{\mathbf{p}^2}{2m} + \frac{m w^2 r^2}{2} - \frac{3\hbar w}{2} + \frac{k^2}{2m} \right) P\psi, \tag{28}$$

where the commutation relation $[r_i, p_j] = \mathrm{i}\hbar \delta_{ij}$ has been used. Thus, we have derived the well-known equation that describes the isotropic harmonic oscillator using a general representation for the matrices β.

The correct result can also be obtained in the vector sector context, applying on equation (27) the operators R^μ and $R^{\mu\nu}$. The harmonic oscillator equation should have a spin–orbit coupling, i.e.

$$E(R^i\psi) = \left(\frac{\mathbf{p}^2}{2m} + \frac{m w^2 r^2}{2} - \frac{3\hbar w}{2} - \frac{w}{\hbar}\mathbf{L}\cdot\mathbf{S} + \frac{k^2}{2m} \right) R^i\psi, \tag{29}$$

where $R^i\psi (i = 1, 2, 3)$ are the components of the vector potential, $\mathbf{L}$ is the orbital angular momentum and $\mathbf{S}$ is the spin-1 operator, such that $(S_m)_{kl} = -\mathrm{i}\hbar\varepsilon_{klm}$ (ε_{klm} being the antisymmetric tensor).

Hence, we have obtained above the appropriate harmonic oscillator equations of the spin-0 and spin-1 sectors, with an essential difference from the one derived in [11]: we have used a general representation for the matrices β and the operators P, P^μ, R^μ and $R^{\mu\nu}$ to select the scalar or vector sectors of the theory. Besides, there is no necessity of performing the non-relativistic limit of the Klein-Gordon equation, as it is the case in [19]. These facts suggest that this formulation generates a consistent treatment to the DKP oscillator in a general projector framework.

3. Gauge invariance and anomalous term

Let us consider that the DKP field in the Galilei-covariant approach interacts with a Galilean electromagnetic field. The minimal coupling generates the following interaction term to be added to the DKP Lagrangian (4),

$$\mathcal{L}_I = e A_\mu \overline{\Psi} \beta^\mu \Psi. \tag{30}$$

The choice of the form of Ψ for the spin-0 sector in this situation requires the change from that given by equation (23) to

$$\Psi = \begin{pmatrix} -\frac{1}{\sqrt{k}} D_\mu \Phi \\ \sqrt{k}\Phi \end{pmatrix}, \tag{31}$$

where $D_\mu = \partial_\mu - \mathrm{i}eA_\mu$ represents the minimal coupling of the system with the non-relativistic eletromagnetic field (A_μ is the five-dimensional potential). Note that this is the right form of the DKP field, in the sense that it gives the consistent local gauge transformations:

$$\Psi \to \Psi' = \mathrm{e}^{\mathrm{i}e\Lambda(x)}\Psi, \qquad \Phi \to \Phi' = e^{\mathrm{i}e\Lambda(x)}\Phi. \tag{32}$$

So, as in the relativistic case, under this local gauge transformation the DKP Lagrangian with the minimal coupling term acquires gauge invariance.

Thus, taking into account the interaction term given by equation (30) in equation (4), the equations of motions become

$$(\beta^\mu D_\mu + k)\Psi = 0, \tag{33}$$

which is the Galilean-covariant minimally coupled DKP equation.

Then, in a similar procedure of the free situation, the independent application of the operators P and P^μ to equation (33) yields

$$(D^\mu D_\mu + k^2)P\Psi = 0. \tag{34}$$

Since the elements of the column matrix $P\Psi$ are scalar fields, as it is explicit in equation (24), then equation (34) acquires the form

$$(D^\mu D_\mu + k^2)\Phi = 0. \tag{35}$$

Equation (35) is the manifestly Galilean-covariant appearance of the Schrödinger equation with minimal coupling. We make this fact clear; first by defining the five-dimensional potential,

$$A_\mu = (\mathbf{A}, -\phi_M, -\phi_E), \tag{36}$$

where $\mathbf{A}$ is the vector potential, ϕ_M and ϕ_E are, respectively, the scalar potentials in the 'magnetic' and in 'electric' limit [20]. Note that they cannot be understood as two simultaneously existing physical scalar potentials. For example, if we want to analyze this model in the magnetic limit, it is retrieved by considering ϕ_E as an auxiliary field, set equal to zero in the equations of motion, whereas ϕ_M is the physical scalar potential field [14]. Then, in the magnetic limit, we can rewrite equation (35) in its known way,

$$i\partial_t\varphi(\mathbf{x}, t) = \left(-\frac{\mathbf{D}^2}{2m} + e\phi_M\right)\varphi(\mathbf{x}, t), \tag{37}$$

where $\mathbf{D} = \nabla - \mathrm{i}e\mathbf{A}$, and we have set $\mathrm{i}\partial_t + k^2 \to \mathrm{i}\partial_t$. Hence, we derived the Schrödinger equation minimally coupled in its well-known form.

Let us now discuss a supposed anomalous term without physical meaning, which appears in the second-order forms obtained from the DKP equation with minimal coupling. Such subject is present in the context of relativistic theory, and here we analyze in the Galilean-covariant scenario. We start by taking the minimally coupled DKP equation, equation (33), and contract it with the operator $D_\alpha\beta^\alpha\beta^\nu$ from the left, giving

$$(\mathrm{i}\beta^\alpha\beta^\nu\beta^\mu D_\alpha D_\mu - k\beta^\alpha\beta^\nu D_\alpha)\Psi = 0. \tag{38}$$

By taking the properties of β matrices in the equation above, it is possible to obtain after some manipulations the expression,

$$D^\nu\Psi = \beta^\alpha\beta^\nu D_\alpha\Psi + \frac{e}{2k}F_{\alpha\mu}\,(\beta^\mu\beta^\nu\beta^\alpha + \beta^\mu g^{\nu\alpha})\,\Psi, \tag{39}$$

where $F_{\mu\nu} = \frac{\mathrm{i}}{e}[D_\mu, D_\nu]$ represents the electromagnetic field strength. Then, we can contract the equation above with the operator D_ν, resulting in the following second-order equation,

$$D_\nu D^\nu\Psi + k^2\Psi - \frac{\mathrm{i}e}{2}F_{\mu\nu}S^{\mu\nu}\Psi - \frac{e}{2k}\,(\beta^\mu\beta^\nu\beta^\alpha + \beta^\mu g^{\nu\alpha})\,D_\nu(F_{\alpha\mu}\Psi) = 0. \tag{40}$$

Thus, like in the relativistic case, in the Galilean-covariant approach we have an anomalous term which is proportional to $\frac{e}{2k}$.

We can prove that this anomalous term has no physical meaning, through the fact that it disappears after the analysis of the physical components of the DKP field in equation (40) (the relativistic situation of this argument was studied in [10]). In the case of spin-0 sector, it is reached by applying the projector P in this second-order equation. Using the result $PS^{\mu\nu} = 0$ (it is natural for the scalar field, $P\Psi$), and after some algebraic manipulations, equation (40) reduces to the right-wave equation, which is equation (34).

Now we discuss the spin-1 sector in a similar way of the case of spin-0 sector, where we have used properly the projectors P and P^{μ}. We apply the operators R^{μ} and $R^{\mu\nu}$ in equation (33), obtaining

$$D_{\alpha}G^{I\alpha\mu} + k^2 R^{\mu}\Psi = 0, \tag{41}$$

where $G^{I\mu\nu} = D^{\mu}R^{\nu}\Psi - D^{\nu}R^{\mu}\Psi$ is the Galilean-covariant stress tensor of the massive vector field $R^{\mu}\Psi$ interacting with a external non-relativistic electromagnetic field. Thus, equation (41) represents the Galilean-covariant version of the minimally coupled Proca equation.

Note also that equation (41) can be rewritten as

$$(D_{\alpha}D^{\alpha} + k^2)R^{\mu}\Psi = 0. \tag{42}$$

The anomalous term in the situation of spin-1 sector can be cleared from the application of the operator R^{λ} from the left of equation (40). After the necessary manipulations, it can be seen that the anomalous term disappears, with equation (40) reducing to the Galilean-covariant minimally coupled Proca equation, given by equation (41).

Hence, we have proved by the use of the projector scenario that, like in the relativistic context of the DKP theory, in the Galilean-covariant approach the anomalous term disappears when the physical components are selected.

4. Concluding remarks

In this paper, we have used the Galilean covariant formalism in five dimensions to analyse the non-relativistic Duffin–Kemmer–Petiau field by a projector formulation. Such approach allows us to select the scalar and vector sectors through the use of the appropriate operators. As an application, we have studied a kind of non-minimal coupling, which is associated to the harmonic oscillator. We have shown that this formulation generates a consistent treatment for the DKP oscillator of scalar and vector bosons in a general projector framework, naturally in the non-relativistic regime and with the selection of the spin-0 and spin-1 sectors without using a particular representation of the β matrices.

We also discussed the local gauge invariance of the model, as well as the anomalous term which appears in the wave equations due the minimal coupling. In both questions our results were carried out as in the relativistic case. We have demonstrated that the consistent local gauge transformations can be obtained through the choice of the right form of the non-relativistic DKP field. Also, the anomalous term disappearance in the second-order equations was derived by making use of the selection of the physical components, which yields the manifestly covariant version of the Schrödinger and vector equations free of spurious terms.

In the sequence, some interesting topics related to the theme studied above deserve investigation. A natural question is about the quantization of the Galilean covariant DKP formalism. Besides, other curious point is the formulation of the Bose–Einstein condensation in this scenario.

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Appendix. Spin 0 representation for the Galilean DKP theory

In this Appendix we explicit a possible choice for a six-dimensional representation of the β matrices:

$$\beta^1 = \begin{pmatrix} 0&0&0&0&0&1\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 1&0&0&0&0&0 \end{pmatrix}, \qquad \beta^2 = \begin{pmatrix} 0&0&0&0&0&0\\ 0&0&0&0&0&1\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&1&0&0&0&0 \end{pmatrix}, \tag{A.1}$$

$$\beta^3 = \begin{pmatrix} 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&1\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&1&0&0&0 \end{pmatrix}, \qquad \beta^4 = \begin{pmatrix} 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&1\\ 0&0&0&0&0&0\\ 0&0&0&0&-1&0 \end{pmatrix}, \tag{A.2}$$

$$\beta^5 = \begin{pmatrix} 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&1\\ 0&0&0&-1&0&0 \end{pmatrix}. \tag{A.3}$$

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